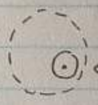
MATB41 Week 4 Notes

1. Open Sets (Continued):

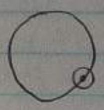
- For an object to be an open set, any arbitrary point within that set must be able to form an open disk or open ball within the open set.

- E.g. 1



← This is an open set.

Any arbitrary point within it is able to form an open disk or open ball.



← Because the open disk is not completely in the set, this is not an open set.

- To prove something is an open set, we have to prove that any arbitrary point within it must be able to form an open disk or open ball.

2. Delta-Epsilon Proofs:

- Informal Definition of Limits:
  Let $a = (a_1, a_2, \ldots a_n)$ be a point in R^n and let $x = (x_1, x_2, \ldots x_n)$ be a point in R^n. Let $f: R^n \to R$, $L \in R$ is called the limit of f as x approaches a if $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.

- This is denoted as
  1. $\lim\limits_{x \to a} f(x) = L$

  2. $\lim\limits_{(x_1, x_2, \ldots, x_n) \to (a_1, a_2, \ldots a_n)} f(x_1, \ldots x_n) = L$, $f(x) \to L$ as $x \to a$

- Formal Definition of Limits:

$\lim_{x \to a} f(x) = L$ if

$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|x - a\| < d$ then $|f(x) - L| < \varepsilon$.

In the case where we have 2 variables, let $\vec{x} = (x, y)$ and $\vec{a} = (a, b)$. Then, if $\sqrt{(x-a)^2 + (y-b)^2} < d$ then $|f(x) - L| < \varepsilon$.

- E.g. 2 Use the definition to prove that

$$\lim_{(x,y) \to (0,0)} xy = 0$$

Magnitude ↓ Abs Value ↓

Solution:

$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|(x,y) - (0,0)\| < d$ then $|xy - 0| < \varepsilon$

$$\|(x,y)\| = \sqrt{x^2 + y^2} < d$$
$$x^2 + y^2 < d^2$$

$$(x+y)^2 = x^2 + 2xy + y^2 \geq 0$$
$$x^2 + y^2 \geq 2xy$$
$$\geq 2|xy|$$
$$\frac{x^2+y^2}{2} \geq |xy|$$

$$|xy| < \varepsilon$$
$$\frac{x^2+y^2}{2} < \frac{d^2}{2}$$

Set $\frac{d^2}{2} = \varepsilon$

$$d = \sqrt{2\varepsilon}$$

Proof:

let $d = \sqrt{2\varepsilon} > 0$

$$|xy| < \frac{x^2+y^2}{2}$$
$$= \frac{d^2}{2}$$
$$= \frac{(\sqrt{2\varepsilon})^2}{2}$$
$$= \varepsilon, \text{ as wanted}$$

$$\therefore \lim_{(x,y)\to(0,0)} xy = 0$$

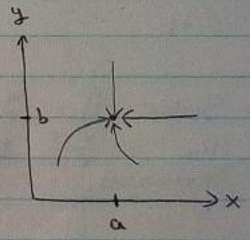
3. Paths of Limits:

- Recall that in single variable limits,
$\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

Here, there are only 2 paths.

- With multi-variable limits, we could approach the point from several paths, as shown below.

$$\lim_{(x,y)\to(a,b)} f(x,y)$$



For a multi-variable limit to exist, the function must be approaching the same value regardless of the path it takes.

I.e. If x approaches a along path A and $f(x) = L$ while x approaches a along path B results in $f(x) = M$ and $L \neq M$, then we say that the limit Does Not Exist (DNE).

- E.g. 3 $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Along the x-axis $(x,y)\to(0,0)$ and y is 0.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-0}{x^2+0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2}$$

$$= 1$$

Along the y-axis $(x,y)\to(0,0)$ and x is 0.

$$\lim_{(x,y)\to(0,0)} \frac{0-y^2}{0+y^2}$$

$$= -1$$

Since the 2 limits don't equal, $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

4. Continuity:

- Informal Definition:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with domain U. Let $x_0 \in U$. We say f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

$\lim_{x \to x_0} f(x) = f(x_0)$ means

1. $x_0 \in$ Domain (f)
2. $\lim_{x \to x_0} f(x) = L$
3. $f(x_0) = L$

If f doesn't satisfy any of these requirements, then f is not cont at x_0.

- Formal Definition:

$\lim_{x \to x_0} f(x) = f(x_0)$ means

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $\|x - x_0\| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

This is very similar to delta-epsilon, but notice that $\|x - x_0\|$ could equal 0 here.

- E.g. 4 Is $f(x,y) = \dfrac{x^2 - y^2}{x+y}$ cont at $(0,0)$?

Solution:

$f(x,y)$ is not cont at $(0,0)$ because there is a hole at $(0,0)$. This means that $(0,0) \notin$ Dom (f).

Note: $(0,0)$ is a removal discontinuity because $\lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x+y} = 0$ (the limit exists), but $f(0,0)$ doesn't.

- E.g. 5. Is $f(x,y) = 3x^2y + \sqrt{xy}$ cont at $(1,2)$?

Solution:

1. $(1,2) \in Dom(f)$

2. $\lim\limits_{(x,y)\to(1,2)} 3x^2y + \sqrt{xy} = 6+\sqrt{2}$

3. $f(1,2) = 6+\sqrt{2}$

$\therefore$ $f(x,y)$ is cont at $(1,2)$.

- E.g. 6 Is $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

cont at $(0,0)$.

Solution:

1. $(0,0) \in Dom(f)$

2. $\lim\limits_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2}$

This leads to nowhere, so we have to use a different method.

3. Along the path $y=x$:

$\frac{(2x)^2}{2x^2} = \frac{4x^2}{2x^2} = 2$

Along the path $y=-x$:

$\frac{0}{2x^2} = 0$

Since we get different results, the limit DNE.

$\therefore$ Not cont at $(0,0)$.

- Properties of Continuity:

1. Let $f(x)$ and $g(x)$ be cont at x_0 and let c be a constant. Then,
   1. $f(x) \pm g(x)$
   2. $cf(x)$
   3. $f(x)g(x)$
   4. $\frac{f(x)}{g(x)}$ | $g(x) \neq 0$

   are cont at x_0.

2. Let $f: U \subset R^n \to R^m$ with $f(x) = (f_1(x), f_2(x), \dots f_m(x))$. f is cont at x_0 iff $f_1, f_2, \dots f_m$ are cont at x_0.

3. Let $g: U_1 \subset R^n \to R^m$.
   Let $f: U_2 \subset R^m \to R^p$.
   Suppose that $g(U_1) \subset U_2$, so that $f \circ g$ is defined on U_1. If g is cont at x_0 and f is cont at $g(x_0)$, then $f \circ g$ is cont at x_0.

4. Trig, polynominal and exponential functions are always cont on their domain.

   If you have a limit, $\lim_{(x,y) \to (a,b)} f(x,y)$, that

5. you know is continuous at (a,b), you can plug (a,b) into $f(x,y)$.

- E.g. 7. Show that $\sin(x+y)$ is cont everywhere in R^2.

Solution: Let $f = \sin(t)$ $R \to R$
Let $g = x+y$ $R^2 \to R$
$f(g)$ is cont everywhere in R^2, so $\sin(x+y)$ is also cont everywhere in R^2.

5. Evaluating Limits:

- Tips:
  1. Pay attention to the degree of the numerator and denominator. Polynomials with higher degrees reach to 0 faster.
  2. We may use $|x| \le \sqrt{x^2+y^2}$ or $|y| \le \sqrt{x^2+y^2}$
  3. In R^2, we may substitute (x,y) with (r,θ).
     $x = r\cos\theta$, $y = r\sin\theta$, $r \ge 0$
  4. We may use the Squeeze Theorem.

- Properties:
  Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ and c be a constant.
  1. $\lim_{x\to a} c = c$
  2. $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x) = cL$
  3. $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
  4. $\lim_{x\to a} (f(x)\,g(x)) = LM$
  5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
  6. $\lim_{x\to a} f(x)^{\frac{m}{n}} = L^{\frac{m}{n}}$, $n \neq 0$

- Eg. 8 Solve $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Solution:

Let $r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2} \to 0$

$\lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1$

- Eg. 9 Solve $\lim_{(x,y)\to(2,8)} 3x^2y + \sqrt{xy}$

$\lim_{(x,y)\to(2,8)} 3x^2y + \sqrt{xy}$

$= \lim_{(x,y)\to(2,8)} 3x^2y + \lim_{(x,y)\to(2,8)} \sqrt{xy}$

$= 3(2)^2(8) + \sqrt{(2)(8)}$

$= 100$

- Eg. 10 Solve $\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

Solution:

$\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

$= \lim_{(x,y)\to(4,1)} \left(\frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}\right)\left(\frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}\right)$

$= \lim_{(x,y)\to(4,1)} \frac{y\cancel{(x-4y)}(\sqrt{x}+2\sqrt{y})}{\cancel{x-4y}}$

$= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y})$

$= (1)(\sqrt{4}+2\sqrt{1})$

$= 4$

- E.g. 11 Prove

a) $\lim\limits_{(x,y)\to(a,b)} x = a$

b) $\lim\limits_{(x,y)\to(a,b)} y = b$

Solution:

a) $\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|(x,y)-(a,b)\| < d$ then $|x-a| < \varepsilon$

$\sqrt{(x-a)^2 + (y-b)^2} < d$

$|x-a| < \sqrt{(x-a)^2 + (y-b)^2} < d$

Choose $d = \varepsilon$

Proof:

Let $d = \varepsilon$

We have $|x-a| < \sqrt{(x-a)^2 + (y-b)^2} < d = \varepsilon$, as wanted

b) $\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|(x,y)-(a,b)\| < d$ then $|y-b| < \varepsilon$

$\sqrt{(x-a)^2 + (y-b)^2} < d$

$|y-b| < \sqrt{(x-a)^2 + (y-b)^2} < d$

Choose $d = \varepsilon$

Proof:

Let $d = \varepsilon$

We have $|y-b| < \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon$, as wanted